Reparametrization of COM-Poisson Regression Models for Analysis of Count Data

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Outline

1. Background
2. Reparametrization
3. Simulation study
4. Case studies
5. Final remarks

1

Background

Poisson model and limitations

Count data

- Number of times an event occurs in the observation unit;
- Mean-variance relationship.

GLM framework (Nelder & Wedderburn 1972)

- Provide suitable distribution for a counting random variables;
- Efficient algorithm for estimation and inference;
- Implemented in many software.

Poisson model

- Relationship between mean and variance, $\text{E}(Y) = \text{Var}(Y)$;

Main limitations

- Overdispersion (more common), $\text{E}(Y) < \text{Var}(Y)$
- Underdispersion (less common), $\text{E}(Y) > \text{Var}(Y)$

Weighted Poisson models

- The family of weighted Poisson distributions (WPD) (**?**), weights the Poisson probability function by a suitable function.

- The probability mass function of the WPD is

$$\Pr(Y = y) = \frac{\exp(-\lambda)\lambda^y}{y!} \frac{w(y)}{\mathrm{E}_\lambda[w(Y)]}, \quad y \in \mathbb{N},$$

  where $\mathrm{E}_\lambda(\cdot)$ denotes the mean value with respect to the Poisson random variable with parameter λ and $w(y)$ is a weight function.

- The weight function may depend on extra parameter to ensure more flexibility to the distribution.

COM-Poisson distribution

- The COM-Poisson (Shmueli et al. 2005) belongs to the family of weighted Poisson distributions with the weight function $w(y, \nu) = (y!)^{1-\nu}$.

- The probability mass function of Y a COM-Poisson random variable is

$$\Pr(Y = y) = \frac{\lambda^y \exp(-\lambda)}{(y!)^\nu \mathrm{E}_\lambda[(Y!)^{1-\nu}]} = \frac{\lambda^y}{(y!)^\nu \mathrm{Z}(\lambda, \nu)}, \quad \mathrm{Z}(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu},$$

  where ν is the dispersion parameter.

- The moments of distribution are not obtained in closed forms. There is approximations proposed by Shmueli et al. (2005), Sellers & Shmueli (2010):

$$\mathrm{E}(Y) \approx \dot{\mathrm{E}}(Y) = \lambda^{1/\nu} - \frac{\nu - 1}{2\nu} \qquad \text{and} \qquad \mathrm{Var}(Y) \approx \frac{\lambda^{1/\nu}}{\nu}.$$

COM-Poisson regression models

Model definition

- Modelling the relationship between $\mathrm{E}(Y_i)$ and $\boldsymbol{x}_i$ indirectly (Sellers & Shmueli 2010);

$$\begin{aligned} Y_i \mid \boldsymbol{x}_i &\sim \text{COM-Poisson}(\lambda_i, \nu) \\ \eta(\mathrm{E}(Y_i \mid \boldsymbol{x}_i)) &= \log(\lambda_i) = \boldsymbol{x}_i^\top \boldsymbol{\beta} \end{aligned}$$

Main goals

- Study distribution properties in terms of i) modelling real count data and ii) inference aspects.
- Propose a reparametrization in order to model the expectation of the response variable as a function of the covariate values directly.

2

Reparametrization

Reparametrized COM-Poisson

Reparametrization

- Introduced new parameter μ, using the mean approximation

$$\mu = \dot{\mathrm{E}}(Y) = \lambda^{1/\nu} - \frac{\nu - 1}{2\nu} \quad \Rightarrow \quad \lambda = \left(\mu + \frac{(\nu - 1)}{2\nu}\right)^{\nu};$$

- Precision parameter is taken on the log scale to avoid restrictions on the parameter space

$$\phi = \log(\nu) \Rightarrow \phi \in \mathbb{R};$$

Probability mass function

- Replacing λ and ν as function of μ and ϕ in the pmf of COM-Poisson

$$\Pr(Y = y \mid \mu, \phi) = \left(\mu + \frac{e^{\phi} - 1}{2e^{\phi}}\right)^{ye^{\phi}} \frac{(y!)^{-e^{\phi}}}{Z(\mu, \phi)}.$$

Study of the moments approximations

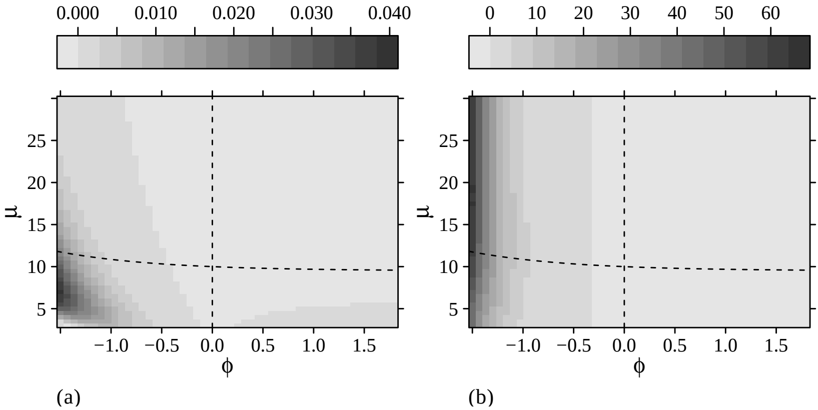


Figure: Quadratic errors for the approximation of the (a) expectation and (b) variance. Dotted lines represent the restriction for suitable approximations given by Shmueli et al. (2005).

COM-Poisson$_\mu$ distribution

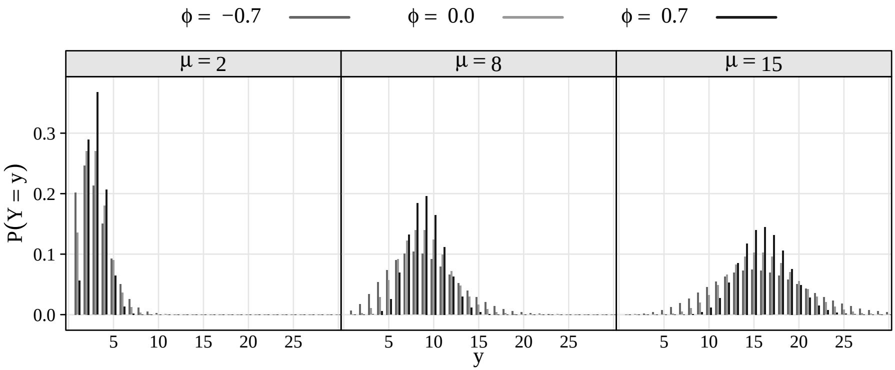


Figure: Shapes of the COM-Poisson distribution for different parameter values.

Properties of COM-Poisson distribution

To explore the flexibility of the COM-Poisson distribution, we consider the follow indexes:

- **Dispersion index:** $\text{DI} = \text{Var}(Y)/\text{E}(Y)$;
- **Zero-inflation index:** $\text{ZI} = 1 + \log \Pr(Y = 0)/\text{E}(Y)$;
- **Heavy-tail index:** $\text{HT} = \Pr(Y = y + 1)/\Pr(Y = y)$, for $y \to \infty$.

These indexes are interpreted in relation to the Poisson distribution:

- over- ($\text{DI} > 1$), under- ($\text{DI} < 1$) and equidispersion ($\text{DI} = 1$);
- zero-inflation ($\text{ZI} > 0$) and zero-deflation ($\text{ZI} < 0$) and
- heavy-tail distribution for $\text{HT} \to 1$ when $y \to \infty$.

Properties of COM-Poisson distribution

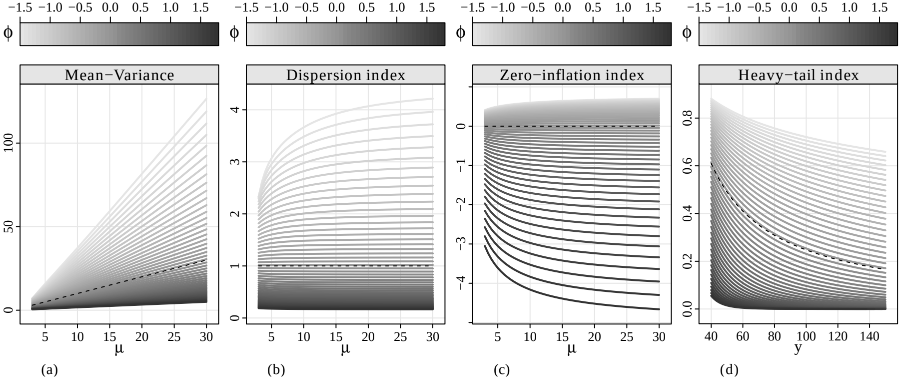


Figure: Indexes for COM-Poisson distribution. (a) Mean and variance relationship, (b–d) dispersion, zero-inflation and heavy-tail indexes for different parameter values. Dotted lines represents the Poisson special case.

COM-Poisson$_\mu$ regression models

Let y_i a set of independent observations from the COM-Poisson and $\boldsymbol{x}_i^\top = (x_{i1}, x_{i2}, \ldots, x_{ip})$ is a vector of known covariates, $i = 1, 2, \ldots, n$.

Model definition

- Modelling relationship between $\dot{\mathrm{E}}(Y_i)$ and $\boldsymbol{x}_i$ directly

$$
\begin{aligned}
Y_i \mid \boldsymbol{x}_i &\sim \text{COM-Poisson}_\mu(\mu_i, \phi) \\
\log(\dot{\mathrm{E}}(Y_i \mid \boldsymbol{x}_i)) &= \log(\mu_i) = \boldsymbol{x}_i^\top \boldsymbol{\beta}
\end{aligned}
$$

Log-likelihood function ($\ell = \ell(\boldsymbol{\beta}, \phi \mid \boldsymbol{y})$)

- $\ell = e^{\phi} \left[\sum_{i=1}^{n} y_i \log\left(\mu_i + \frac{e^{\phi} - 1}{2e^{\phi}}\right) - \sum_{i=1}^{n} \log(y_i!) \right] - \sum_{i=1}^{n} \log(Z(\mu_i, \phi))$

where $\mu_i = \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})$

Estimation and inference

The estimation and inference is based on the method of maximum likelihood. Let $\boldsymbol{\theta} = (\boldsymbol{\beta}, \phi)$ the model parameters.

- Parameter estimates are obtained by numerical maximization of the log-likelihood function (by BFGS algorithm);
  $\ell(\hat{\boldsymbol{\theta}}) = \max \ell(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \mathbb{R}^{p+1}$;
- Standard errors for regression coefficients are obtained based on the observed information matrix;
  $\text{Var}(\hat{\boldsymbol{\theta}}) = -\mathcal{H}^{-1}$, where $\mathcal{H}$ is the matrix of second partial derivatives at $\hat{\boldsymbol{\theta}}$;
- Confidence intervals for $\hat{\mu}_i$ are obtained by delta method.
  $\text{Var}[g(\hat{\boldsymbol{\theta}})] \doteq \boldsymbol{G}\,\text{Var}(\hat{\boldsymbol{\theta}})\boldsymbol{G}^\top$, where $\boldsymbol{G}^\top = (\partial g/\partial\beta_1, \ldots, \partial g/\partial\beta_p)^\top$;
- The Hessian matrix $\mathcal{H}$ is obtained numerically by finite differences.

3

Simulation study

Definitions on the simulation study

Objective: assess the properties of maximum likelihood estimators and orthogonality in the reparametrized model;

Simulation: we consider counts generated according a regression model with a continuous and categorical covariates and different dispersion scenarios.

Algorithm 1: Steps in simulation study.

for $n \in \{50, 100, 300, 1000\}$ **do**
    set $\boldsymbol{x}_1$ as a sequence, with n elements, between 0 and 1;
    set $\boldsymbol{x}_2$ as a repetition, with n elements, of three categories;
    compute $\boldsymbol{\mu}$ using $\boldsymbol{\mu} = \exp(\beta_0 + \beta_1 \boldsymbol{x}_1 + \beta_{21} \boldsymbol{x}_{21} + \beta_{22} \boldsymbol{x}_{22})$;
    for $\phi \in \{-1.6, -1.0, 0.0, 1.8\}$ **do**
        repeat
            simulate $\boldsymbol{y}$ from COM-Poisson distribution with $\boldsymbol{\mu}$ and ϕ parameters;
            fit COM-Poisson$_\mu$ regression model to simulated $\boldsymbol{y}$;
            get $\hat{\boldsymbol{\theta}} = (\hat{\phi},\ \hat{\beta}_0,\ \hat{\beta}_1,\ \hat{\beta}_{21},\ \hat{\beta}_{22})$;
            get confidence intervals for $\hat{\boldsymbol{\theta}}$ based on the observed information matrix.
        until 1000 *times*;

Definitions on the simulation study

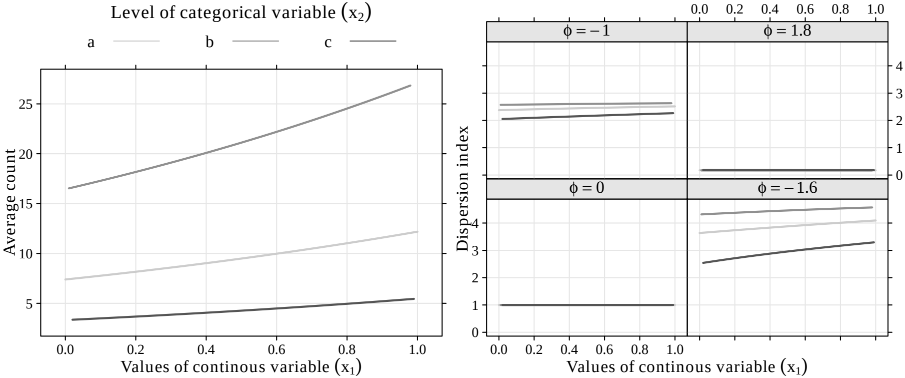
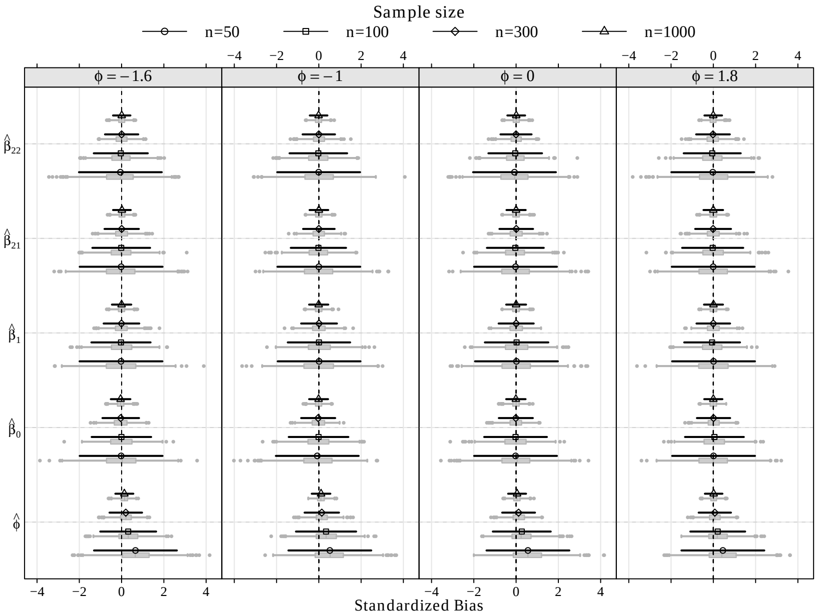


Figure: Average counts (left) and dispersion indexes (right) for each scenario considered in the simulation study.

Bias of the estimators



Coverage rate of the confidence intervals

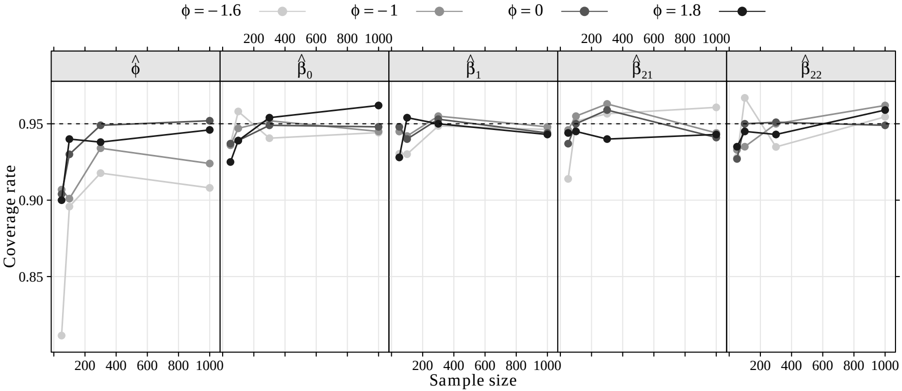


Figure: Coverage rate based on confidence intervals obtained by quadratic approximation for different sample sizes and dispersion levels.

Orthogonality property of the MLEs

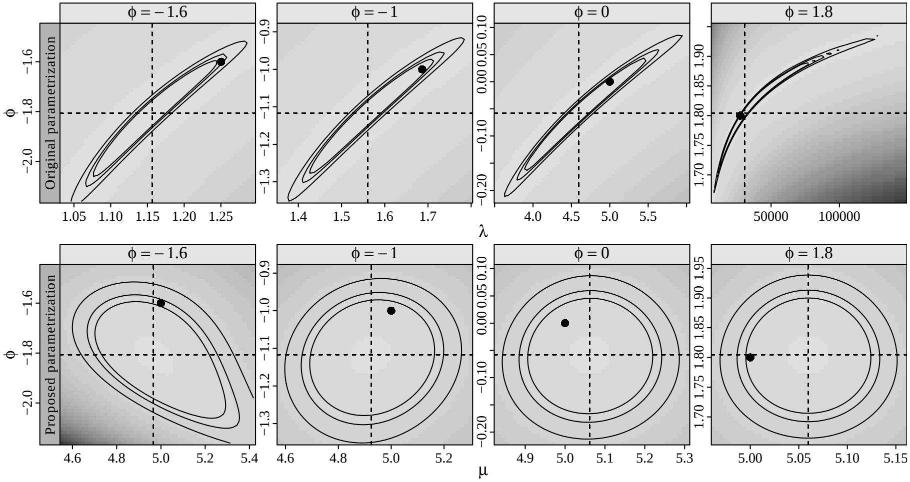


Figure: Deviance surfaces contour plots under original and proposed parametrization. The ellipses are confidence regions (90, 95 and 99%), dotted lines are the maximum likelihood estimates, and points are the real parameters used in the simulation.

4

Case studies

Motivating data sets and data analysis

- Three illustrative examples of count data analysis are reported.
  - Assessing toxicity of nitrofen in aquatic systems, an equidispersed example;
  - Soil moisture and potassium doses on soybean culture, an overdispersed example; and
  - Artificial defoliation in cotton phenology, an underdispersed example.
- In the data analysis, we consider:
  - The COM-Poisson (original parametrization) model;
  - The COM-Poisson (new parametrization) model;
  - Quasi-Poisson model ($\text{Var}(Y) = \sigma \text{E}(Y)$); and
  - The standard Poisson regression model.

4.1

Case studies

Artificial defoliation in cotton phenology

Cotton bolls data

Aim: to assess the effects of five defoliation levels on the bolls produced at five growth stages;

Design: factorial 5×5, with 5 replicates;

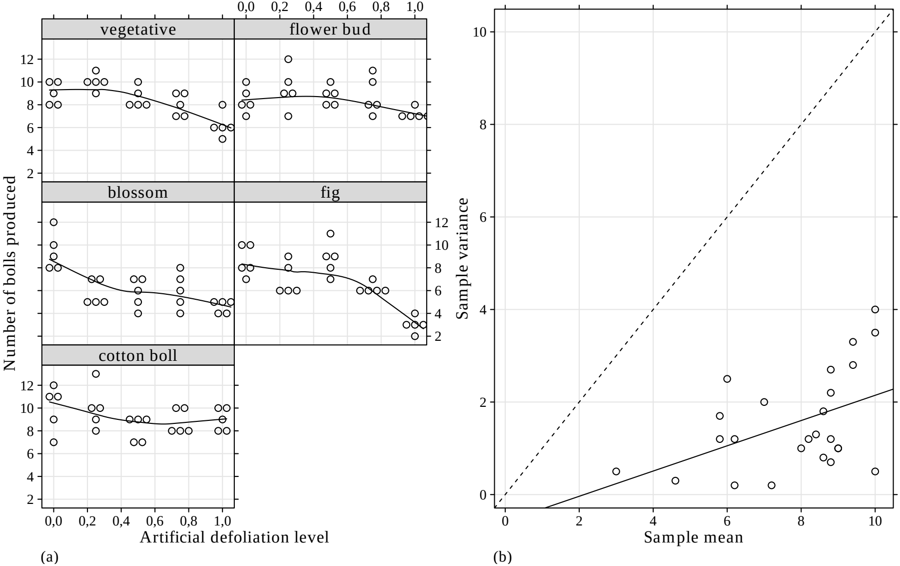
Experimental unit: a plot with 2 plants;

Factors:

- Artificial defoliation (`des`):
- Growth stage (`est`):

Response variable: Total number of cotton bolls;

Cotton bolls data



Model specification

Linear predictor: following Zeviani et al. (2014)

- $\log(\mu_{ij}) = \beta_0 + \beta_{1j}\text{def}_i + \beta_{2j}\text{def}_i^2$
  i varies in the levels of artificial defoliation;
  j varies in the levels of growth stages.

Alternative models:

- Poisson (μ_{ij});
- COM-Poisson ($\lambda_{ij} = \eta(\mu_{ij})\,,\, \phi$)
- COM-Poisson$_\mu$ ($\mu_{ij}\,,\, \phi$)
- Quasi-Poisson ($\text{Var}(Y_{ij}) = \sigma\mu_{ij}$)

Parameter estimates

Table: Parameter estimates (Est) and ratio between estimate and standard error (SE).

	Poisson		COM-Poisson		COM-Poisson$_\mu$		Quasi-Poisson	
	Est	Est/SE	Est	Est/SE	Est	Est/SE	Est	Est/SE
ϕ , σ			1.585	12.417	1.582	12.392	0.241	
β_0	2.190	34.572	10.897	7.759	2.190	74.640	2.190	70.420
β_{11}	0.437	0.847	2.019	1.770	0.435	1.819	0.437	1.726
β_{12}	0.290	0.571	1.343	1.211	0.288	1.223	0.290	1.162
β_{13}	−1.242	−2.058	−5.750	−3.886	−1.247	−4.420	−1.242	−4.192
β_{14}	0.365	0.645	1.595	1.298	0.350	1.328	0.365	1.314
β_{15}	0.009	0.018	0.038	0.035	0.008	0.032	0.009	0.036
β_{21}	−0.805	−1.379	−3.725	−2.775	−0.803	−2.961	−0.805	−2.809
β_{22}	−0.488	−0.861	−2.265	−1.805	−0.486	−1.850	−0.488	−1.754
β_{23}	0.673	0.989	3.135	2.084	0.679	2.135	0.673	2.015
β_{24}	−1.310	−1.948	−5.894	−3.657	−1.288	−4.095	−1.310	−3.967
β_{25}	−0.020	−0.036	−0.090	−0.076	−0.019	−0.074	−0.020	−0.074
LogLik	−255.803		−208.250		−208.398		–	
AIC	533.606		440.500		440.795		–	
BIC	564.718		474.440		474.735		–	

Fitted curves

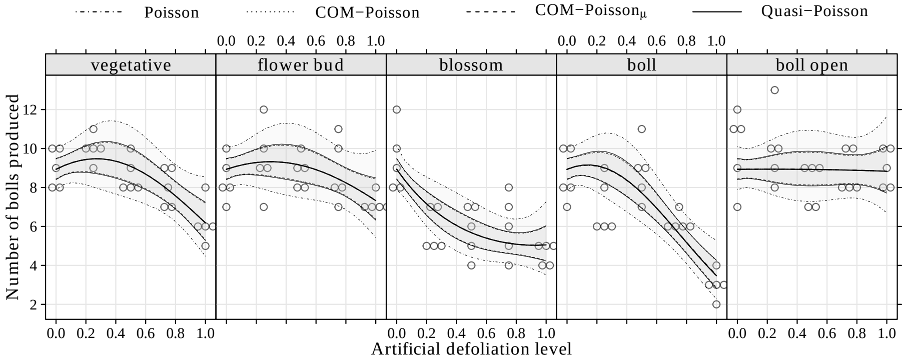


Figure: Scatterplots of the observed data and curves of fitted values with 95% confidence intervals as functions of the defoliation level for each growth stage.

4.2

Case studies

Soil moisture and potassium doses on soybean culture

Soybean data

Aim: evaluate the effects of potassium doses applied to soil in different soil moisture levels;

Design: factorial 5×3 in a randomized complete block design (5 blocks);

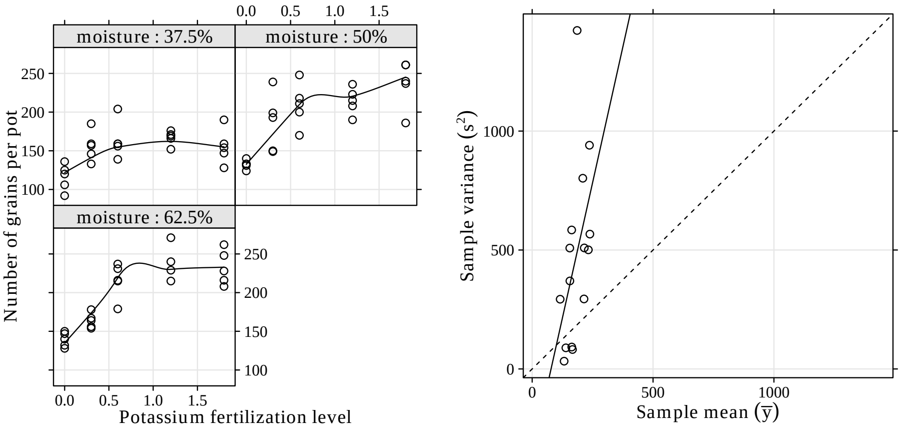
Experimental unit: a pot with a plant;

Factors:

- Potassium fertilization dose (`K`):
- Soil moisture level (`umid`):

Response variable: Total number of bean seeds per pot;

Soybean data



Model specification

Linear predictor: based on descriptive analysis,

- $\log(\mu_{ijk}) = \beta_0 + \gamma_i + \tau_j + \beta_1 \mathsf{K}_k + \beta_2 \mathsf{K}_k^2 + \beta_{3j} \mathsf{K}_k$
  i varies according the blocks;
  j varies in the levels of soil moisture;
  k varies in the levels of potassium fertilization.

Alternative models:

- Poisson (μ_{ij});
- COM-Poisson ($\lambda_{ij} = \eta(\mu_{ij})\,,\,\phi$)
- COM-Poisson$_\mu$ ($\mu_{ij}\,,\,\phi$)
- Quasi-Poisson ($\text{var}(Y_{ij}) = \sigma\mu_{ij}$)

Parameter estimates

Table: Parameter estimates (Est) and ratio between estimate and standard error (SE).

	Poisson		COM-Poisson		COM-Poisson$_\mu$		Quasi-Poisson	
	Est	Est/SE	Est	Est/SE	Est	Est/SE	Est	Est/SE
ϕ, σ			−0.779	−4.721	−0.782	−4.737	2.615	
β_0	4.867	144.289	2.232	6.042	4.867	97.781	4.867	89.225
γ_1	−0.019	−0.730	−0.009	−0.494	−0.019	−0.495	−0.019	−0.452
γ_2	−0.037	−1.373	−0.017	−0.921	−0.037	−0.931	−0.037	−0.849
γ_3	−0.106	−3.889	−0.049	−2.422	−0.106	−2.634	−0.106	−2.405
γ_4	−0.092	−3.300	−0.042	−2.102	−0.092	−2.237	−0.092	−2.040
τ_1	0.132	3.647	0.061	2.295	0.132	2.472	0.132	2.255
τ_2	0.124	3.432	0.057	2.177	0.124	2.326	0.124	2.122
β_1	0.616	11.014	0.284	4.729	0.616	7.464	0.616	6.811
β_2	−0.276	−10.250	−0.127	−4.589	−0.276	−6.946	−0.276	−6.338
β_{31}	0.146	4.268	0.067	2.614	0.146	2.892	0.146	2.639
β_{32}	0.165	4.829	0.076	2.884	0.165	3.272	0.165	2.986
LogLik	−340.082		−325.241		−325.233		–	
AIC	702.164		674.482		674.467		–	
BIC	727.508		702.130		702.116		–	

Fitted curves

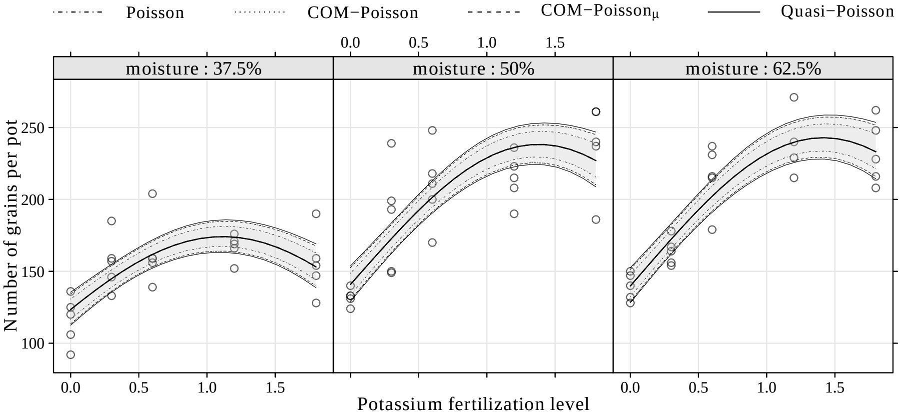


Figure: Dispersion diagrams of been seeds counts as function of potassium doses and humidity levels with fitted curves and confidence intervals (95%).

4.3

Case studies

Assessing toxicity of nitrofen in aquatic systems

Nitrofen data

Aim: measure the reproductive toxicity of the herbicide nitrofen on a species of zooplankton (*Ceriodaphnia dubia*);

Design: completely randomized design, with 10 replicates;

Experimental unit: zooplankton animal;

Factors:

- herbicide nitrofen dose (`dose`);

Response variable: Total number of live offspring;

Model specification

Linear predictors:

$$
\begin{array}{ll}
\text{Linear:} & \log(\mu_i) = \beta_0 + \beta_1 \mathsf{dose}_i, \\
\text{Quadratic:} & \log(\mu_i) = \beta_0 + \beta_1 \mathsf{dose}_i + \beta_2 \mathsf{dose}_i^2 \text{ and} \\
\text{Cubic:} & \log(\mu_i) = \beta_0 + \beta_1 \mathsf{dose}_i + \beta_2 \mathsf{dose}_i^2 + \beta_3 \mathsf{dose}_i^3.
\end{array}
$$

Alternative models:

- Poisson (μ_{ij});
- COM-Poisson ($\lambda_{ij} = \eta(\mu_{ij})$, ϕ)
- COM-Poisson$_\mu$ (μ_{ij} , ϕ)
- Quasi-Poisson ($\text{var}(Y_{ij}) = \sigma\mu_{ij}$)

Likelihood ratio tests

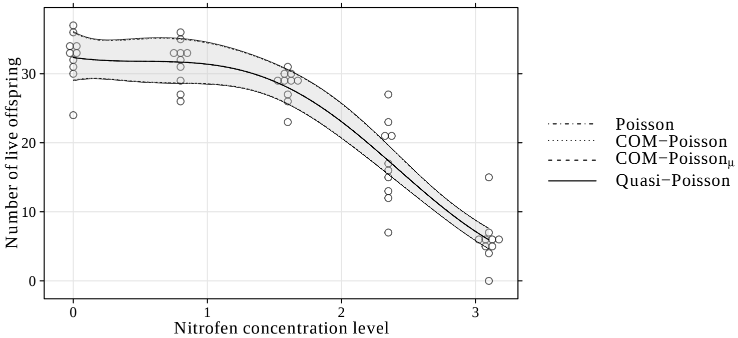
Table: Model fit measures and comparisons between linear predictors.

Poisson	np	ℓ	AIC	2(diff ℓ)	diff np	$P(>\chi^2)$	
Linear	2	−180.667	365.335				
Quadratic	3	−147.008	300.016	67.319	1	2.31E−16	
Cubic	4	−144.090	296.180	5.835	1	1.57E−02	
COM-Poisson	np	ℓ	AIC	2(diff ℓ)	diff np	$P(>\chi^2)$	$\hat{\phi}$
Linear	3	−167.954	341.908				−0.893
Quadratic	4	−146.964	301.929	41.980	1	9.22E−11	−0.059
Cubic	5	−144.064	298.129	5.800	1	1.60E−02	0.048
COM-Poisson$_\mu$	np	ℓ	AIC	2(diff ℓ)	diff np	$P(>\chi^2)$	$\hat{\phi}$
Linear	3	−167.652	341.305				−0.905
Quadratic	4	−146.950	301.900	41.405	1	1.24E−10	−0.069
Cubic	5	−144.064	298.127	5.773	1	1.63E−02	0.047
Quasi-Poisson	np	QDev	AIC	F	diff np	$P(>F)$	$\hat{\sigma}$
Linear	3	123.929					2.262
Quadratic	4	56.610		60.840	1	5.07E−10	1.106
Cubic	5	50.774		5.659	1	2.16E−02	1.031

Parameter estimates and fitted values

Table: Parameter estimates (Est) and ratio between estimate and standard error (SE).

	Poisson		COM-Poisson		COM-Poisson$_\mu$		Quasi-Poisson	
	Est	Est/SE	Est	Est/SE	Est	Est/SE	Est	Est/SE
β_0	3.477	62.817	3.649	4.850	3.477	64.308	3.477	61.860
β_1	−0.086	−0.433	−0.091	−0.448	−0.088	−0.452	−0.086	−0.426
β_2	0.153	0.863	0.161	0.878	0.155	0.894	0.153	0.850
β_3	−0.097	−2.398	−0.102	−2.229	−0.098	−2.464	−0.097	−2.361



4.4
Case studies

Additional results

To compare the computational times on the two parametrizations we repeat the fitting 50 times.

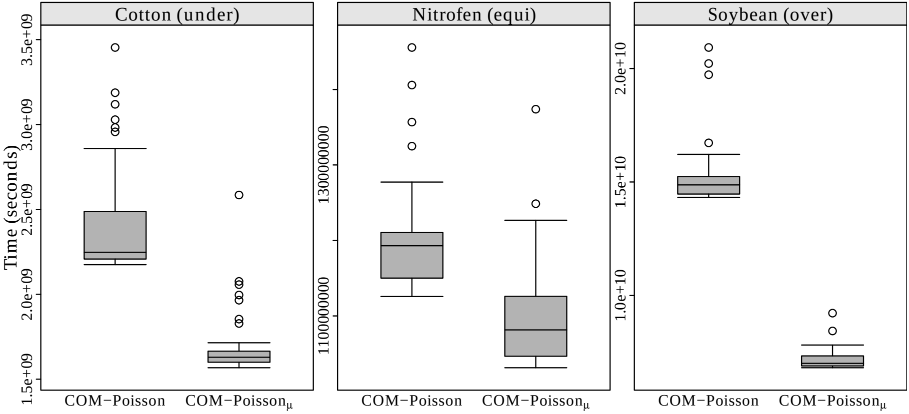


Figure: Computational times to fit the models under original and reparametrized versions based on the fifty repetitions.

5

Final remarks

Concluding remarks

Summary

- Over/under-dispersion needs caution;
- COM-Poisson is a suitable choice for these situations;
- The proposed reparametrization, COM-Poisson$_\mu$ has some advantages:
  - Simple transformation of the parameter space;
  - Leads to the orthogonality of the parameters (seen empirically);
  - Full parametric approach;
  - Empirical correlation between the estimators was practically null;
  - Faster for fitting;
  - Allows interpretation of the coefficients directly (like GLM-Poisson model).

Future work

- Simulation study to assess model robustness against distribution miss specification;
- Assess theoretical approximations for $Z(\lambda, \nu)$ (or $Z(\mu, \phi)$), in order to avoid the selection of sum's upper bound;
- Propose a mixed GLM based on the COM-Poisson$_\mu$ model.

- Full-text article is available on arXiv
  https://arxiv.org/abs/1801.09795
- All codes (in R) and source files are available on GitHub
  https://github.com/jreduardo/article-reparcmp

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